Relations

- Relations: Definition and Notation
- Properties of Relations
- Combining Relations
- Operations on Relations: Projection and Join
- Equivalence Relations and Equivalence Classes
- Partial Order

Imdad ullah Khan

Relations on a Set

Relation

A (binary) relation from X to Y is a subset of $X \times Y$

Relation on a Set

A (binary) relation on a set X is a subset of $X \times X$ (relation from X to X)

Properties of Relations

Reflexive

A relation R on a set X is reflexive if $(a, a) \in R$ for every element $a \in X$

$$A = \{1, 2, 3, 4\}$$

$\boxed{\textbf{ICP 6-5}}$ Which of the following relations are reflexive ?

- $R_1 = \{(1,1),(1,2),(2,3),(3,3),(4,4)\}$ ▷ **No**
- $R_2 = \{(1,1),(2,2),(2,3),(3,3),(4,4)\}$ ▷ **Yes**
- $R_3 = \{(1,1),(2,2),(3,3)\}$ ▷ **No**

Properties of Relations

Symmetric

A relation R on a set X is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in X$

$$A = \{1, 2, 3, 4\}$$

$\boxed{\textbf{ICP 6-6}}$ Which of the following relations are symmetric ?

- $R_1 = \{(1,1), (1,2), (2,1), (3,3), (4,4)\}$ ▷ **Yes**
- $R_2 = \{(1,1)\}$ ▷ **Yes**
- $R_3 = \{(1,3), (3,2), (2,1)\}$ ▷ **No**

Properties of Relations

Antisymmetric

A relation R on a set X is antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$

$$A = \{1, 2, 3, 4\}$$

ICP 6-7 Which of the following relations are antisymmetric ?

- $R_1 = \{(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)\}$ ▷ **No**
- $R_2 = \{(1, 1)\}$ ▷ **Yes**
- $R_3 = \{(1, 3), (3, 2), (2, 1)\}$ ▷ **Yes**

A relation can be symmetric, antisymmetric, both or none

Properties of Relations

Symmetric

A relation R on a set X is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in X$

Antisymmetric

A relation R on a set X is antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$

ICP 6-8 Let $X = \{a, b, c, d\}$. Construct a relation on X that is

1. Symmetric and Antisymmetric
2. Symmetric but not Antisymmetric
3. Not Symmetric but Antisymmetric
4. Not Symmetric and not Antisymmetric

Properties of Relations

Transitive

A relation R on a set X is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

$$A = \{1, 2, 3, 4\}$$

ICP 6-9 Which of the following relations are transitive ?

- $R_1 = \{(1,1), (1,2), (2,2), (2,1), (3,3)\}$ ▷ **Yes**
- $R_2 = \{(1,3), (3,2), (2,1)\}$ ▷ **No**
- $R_3 = \{(2,4), (4,3), (2,3), (4,1)\}$ ▷ **No**

Relations on the set of integers

- $R_1 = \{(a,b) \mid a \leq b\}$
- $R_2 = \{(a,b) \mid a > b\}$
- $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
- $R_4 = \{(a,b) \mid a = b\}$
- $R_5 = \{(a,b) \mid a = b + 1\}$
- $R_6 = \{(a,b) \mid a + b \leq 3\}$

ICP 6-10 Check if the relation has the given property

	R_1	R_2	R_3	R_4	R_5	R_6
reflexive	✓	✗				
symmetric	✗	✗				
anitsymmetric	✓	✓				
transitive	✓	✓				

Representing Relations

$$A = \{a_1, a_2, \ldots a_m\} \quad \text{and} \quad B = \{b_1, b_2, \ldots, b_n\}$$

A relation R from A to B is represented by a

$$m \times n \quad \text{Boolean matrix} \quad M_R = [m_{ij}]$$

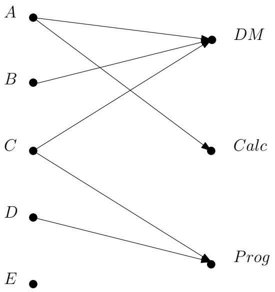
- One row for each element of A
- One column for each element of B

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Representing Relations

$$X = \underbrace{\{A, B, C, D, E\}}_{Students} \qquad \underbrace{Y = \{Calc, DM, Prog\}}_{Courses}$$

$$R := \{(A, DM), (A, Calc), (B, DM), (C, DM), (C, Prog), (D, Prog)\}$$



$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Representing Relations

Relation on a set is represented by a square matrix

$$A = \{1, 2, 3, 4, 6\}$$

$$R := \{(x, y) | x \text{ divides } y\}$$

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Representing Relations

$$A = \{1, 2, 3, 4\}$$

$\boxed{\textbf{ICP 6-11}}$ Represent the relation Q as a matrix.

$$Q = \{(1,1), (1,2), (2,2), (2,1), (3,3)\}$$

Visualizing Properties of Relations

How does M_R look like when R is reflexive?

$$M_R = \begin{pmatrix} \boxed{1} & * & * & * & * & * \\ * & \boxed{1} & * & * & * & * \\ * & * & \boxed{1} & * & * & * \\ * & * & * & \boxed{1} & * & * \\ * & * & * & * & \boxed{1} & * \\ * & * & * & * & * & \boxed{1} \end{pmatrix}$$

Visualizing Properties of Relations

How does M_R look like when R is symmetric?

$$M_R = \begin{pmatrix} * & 0 & 1 & 0 & 1 & 1 \\ 0 & * & 1 & 0 & 1 & 0 \\ 1 & 1 & * & 0 & 0 & 1 \\ 0 & 0 & 0 & * & 0 & 0 \\ 1 & 1 & 0 & 0 & * & 1 \\ 1 & 0 & 1 & 0 & 1 & * \end{pmatrix}$$

M_R is symmetric